Euler class of oriented real vector bundles

Goal: $E \to B$ oriented real rank n vector bundle $\rightsquigarrow$ $e(E) \in H^n(B, \mathbb{Z})$.

- An orientation of a vector space V = choice of one of two equivalence classes of bases where $\sim$ if change of basis matrix has det >0.

  $\Leftrightarrow$ choose an orientation of $\Lambda^{top} V$

  $\Leftrightarrow$ choice of a generator of $H_n(V, V_0; \mathbb{Z}) \underset{\uparrow}{\simeq} \mathbb{Z}$ where $V_0 = V \setminus \{0\}$

  noncanonical, becomes canonical once V oriented!

  (then $H^n(V, V_0; \mathbb{Z}) \simeq \mathbb{Z}$ also has a preferred generator u_V).

Def: orientation of a vector bundle $E \to B$ = orientation of each fiber, consistently (ie. continuously over B wrt local triv's.).

ie. the generators $u_F \in H^n(F, F_0; \mathbb{Z})$ fit locally over $U \subset B$ trivialization charts $\rightsquigarrow u \in H^n(\pi^{-1}(U), \pi^{-1}(U)_0; \mathbb{Z})$

- Thom isomorphism theorem:

Thm: $E_0 = E \setminus B$ complement of zero section

$E \xrightarrow{\pi} B$ oriented rk n $\Rightarrow$ $H^i(E, E_0; \mathbb{Z}) = 0$ for $i<n$, and $\exists!\, u \in H^n(E, E_0; \mathbb{Z})$ st. $\forall$ fiber F, $u_{|F} = u_F \in H^n(F, F_0; \mathbb{Z})$, called the Thom class of E.

Moreover, $\forall k \geq 0$, $H^k(B, \mathbb{Z}) \longrightarrow H^{n+k}(E, E_0; \mathbb{Z})$ is an isom.

$$\alpha \longmapsto (\pi^* \alpha) \cup u$$

Def: The Euler class $e(E)$ is the image of the Thom class under

$$H^n(E, E_0; \mathbb{Z}) \longrightarrow H^n(E, \mathbb{Z}) \xleftarrow[\pi^*]{\sim} H^n(B, \mathbb{Z})$$

Idea pf: 1) If $E = \mathbb{R}^n \times B \to B$ trivial bundle then $H^*(E, E_0) = H^*(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0))$
$\cong H^{*-n}(B)$

(isom $H^k(B) \xrightarrow{\sim} H^{n+k}(B \times \mathbb{R}^n, B \times (\mathbb{R}^n - 0))$
$\alpha \longmapsto \alpha \times e$, e generator of $H^n(\mathbb{R}^n, \mathbb{R}^n - 0)$)

2) $B = B_1 \cup B_2$, result true for $E|_{B_1}, E|_{B_2}, E|_{B_1 \cap B_2} =: E_1, E_2, E_{12}$

Then Mayer-Vietoris

$$\dots \to H^{i-1}(E_{12}, E_{12}^0) \to H^i(E, E^0) \to H^i(E_1, E_1^0) \oplus H^i(E_2, E_2^0) \to H^i(E_{12}, E_{12}^0) \to \dots$$

$\Rightarrow$ for $i<n$, $H^i(E, E^0) = 0$; for $i=n$, $u_1, u_2 \in H^n(\dots)$ have $u_1|_{E_{12}} = u_2|_{E_{12}} = u_{12}$

(by charact. property of restr. to fiber) so $\exists!\, u$ mapping to (u_1, u_2)
and $u_{|\text{fiber}} = u_{i|\text{fiber}}$ ✓

Moreover,

$$\begin{array}{ccccccc} \cdots \to & H^{i-1}(B_{12}) & \to & H^i(B) & \to & H^i(B_1)\oplus H^i(B_2) & \to \cdots \\ & \downarrow \cup\, u_{12} & & \downarrow \cup\, u & & \downarrow (\cup u_1, \cup u_2) & \\ \cdots & H^{n+i-1}(E_{12}, E_{12}^0) & \to & H^{n+i}(E, E^0) & \to & H^{n+i}(E_1, E_1^0)\oplus H^{n+i}(E_2, E_2^0) & \cdots \end{array}$$

Commutative diagram (since restriction maps ...) $\Rightarrow$ by five lemma, isom.

3) Case of compact B (or covered by finitely many triv. charts) by induction.

General case: more work (can't just do $\varinjlim$ over inclusions because $\mathbb{Z}$ not field).

<u>Better viewpoint</u>: <u>spectral sequences</u>. Relative Leray-Serre sequence for (E, E_0) will produce a spectral seq. with $E_2^{pq} = H^p(B; \{H^q(E_x, E_x\setminus 0; \mathbb{Z})\})$
local coeffs cohomology

and cv to $H^{p+q}(E, E\setminus 0; \mathbb{Z})$

- orientation of $E \Rightarrow$ canonically $H^q = \mathbb{Z}$ for $q=n$; 0 else, so $E_2^{pn} = H^p(B, \mathbb{Z})$
others zero
- so s.s. degenerates and gives $H^{n+p}(E, E\setminus 0) \cong H^p(B, \mathbb{Z})$.

$\blacktriangle$

<u>Properties of Euler class:</u> $e(E) \in H^n(B, \mathbb{Z})$:

- <u>Naturality</u>: $\| \; e(f^*E) = f^* e(E)$.

  follows from naturality of Thom class under
  $$\begin{array}{ccc} (f^*E, (f^*E)_0) & \to & (E, E_0) \\ \downarrow & & \downarrow \pi \\ A & \xrightarrow{f} & B \end{array}$$

- <u>Orientation</u>: $\|$ if $\bar{E} = E$ with reversed orientation, $e(\bar{E}) = -e(E)$.

  (since Thom class of $\bar{E}$ is $\bar{u} = -u$)

- $\|$ If $n = \operatorname{rank}(E)$ is odd then $2e(E) = 0$.
  (since $-id: E \to E$ orient.-reversing automorphism $\Rightarrow E \simeq \bar{E}$)

  So, more interesting for $\operatorname{rank}(E)$ even ...

- <u>Direct sum</u>: $\| \; e(E \oplus E') = e(E) \cup e(E')$. (orient.: basis of E then of E')

<u>Pf</u>: • first consider $\begin{array}{c} E\times E' \\ \downarrow \\ B\times B' \end{array}$ and show $e(E\times E') = e(E)\times e(E') \in H^{n+n'}(B\times B', \mathbb{Z})$
$(\overset{\text{def}}{=} p_1^* e(E) \cup p_2^* e(E'))$

Indeed, cross classes behave well under

$$H^n(\mathbb{R}^n, \mathbb{R}^n - 0) \times H^{n'}(\mathbb{R}^{n'}, \mathbb{R}^{n'} - 0) \xrightarrow[\times]{} H^{n+n'}(\mathbb{R}^{n+n'}, \underbrace{(\mathbb{R}^n - 0)\times \mathbb{R}^{n'} \cup \mathbb{R}^n \times (\mathbb{R}^{n'} - 0)}_{= \mathbb{R}^{n+n'} - 0})$$

$\Rightarrow$ Thom classes satisfy $u(E\times E') = (-1)^{nn'} u(E) \times u(E') \in H^{n+n'}(E\times E', (E\times E')_0)$

$\Rightarrow$ so do images in $H^{n+n'}(E\times E') \simeq H^{n+n'}(B\times B')$, ie. $e(E\times E') = (-1)^{nn'} e(E)\times e(E')$

Sign is not important since $e = -e$ when either n or n' odd.

- Finally: recall diagonal restriction

$$\begin{array}{ccc} E\oplus E' & \to & E\times E' \\ \downarrow & & \downarrow \\ \delta: B & \to & B\times B \\ b & \mapsto & (b,b) \end{array}$$

so $e(E\oplus E') = \delta^* e(E\times E') = \delta^*(e(E)\times e(E')) = e(E) \cup e(E')$

$(\delta^* p_i^* = id^* = id)$. ▲

- <u>Prop:</u> ‖ if E has a nowhere vanishing section then $e(E) = 0$.

<u>Pf.</u> given such a section s, $s: B \to E_0$, and the composition $B \xrightarrow{s} E_0 \xrightarrow{incl.} E \xrightarrow{\pi} B$ is identity.

so $H^n(B) \xrightarrow{\pi^*} H^n(E) \longrightarrow H^n(E_0) \xrightarrow{s^*} H^n(B)$ is identity

but $e(E) \longmapsto u(E) \longmapsto 0$ hence $e(E) = 0$. ▲

(Alternative proof: assuming E has a Eucl. metric, can split $E = L \oplus L^\perp$, $L \subset E$ span of s = trivial line subbundle, $e(L) = 0$, and $e(E) = e(L) \cup e(L^\perp) = 0$.)

<u>Alternative def. of Stiefel-Whitney classes:</u> [Charles Rezk on mathoverflow] (not $\mathbb{Z}$-oriented)

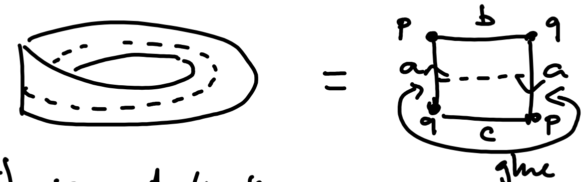
- can imitate the above with $\mathbb{Z}/2$ coeffs to define, for any real v.b., V a mod 2 Thom class $\bar{u} \in H^n(E, E_0; \mathbb{Z}/2)$ characterized by $\bar{u}|_{fiber}$ = generator of $H^n(F, F_0; \mathbb{Z}/2) \simeq \mathbb{Z}/2$

and a mod 2 Euler class $\bar{e}(E) \in H^n(B, \mathbb{Z}/2)$ ($\pi^* \bar{e}(E)$ image of $\bar{u}(E)$ in $H^n(E, \mathbb{Z}/2)$)

- This still has all the properties above (naturality, $\oplus$: product)

Lemma: for $\tau \to \mathbb{RP}^1$, $\bar{e}(\tau) =$ generator of $H^1(\mathbb{RP}^1, \mathbb{Z}/2)$

Pf: $\tau \simeq$ Möbius strip,



and the Thom class $u(\tau)$ is rep^d by the 1-cocycle $a \mapsto 1$, $b, c \mapsto 0$.
in $H^1(\text{Möbius}, \mathbb{Z}/2)$ this is cohomologous to $a \mapsto 0$, $b, c \mapsto 1$
(by adding δ of 0-cocycle $p \mapsto 0$, $q \mapsto 1$) ie. $\pi^*(\text{gen.})$. $\blacktriangle$

- given $E \to B$, consider $E \times S^\infty \to B \times S^\infty$, and produce a vector bundle $\tilde{E} \to B \times \mathbb{RP}^\infty$ by quotienting by involutions $(b, x) \mapsto (b, -x)$ on $B \times S^\infty$
$(v, x) \mapsto (-v, -x)$ on $E \times S^\infty$

(notes: $\tilde{E} \cong p_1^* E \otimes p_2^* \tau$).

Then $\underset{\text{degree } n}{\bar{e}(\tilde{E})} \in H^*(B \times \mathbb{RP}^\infty, \mathbb{Z}/2) \simeq H^*(B, \mathbb{Z}/2)[t]$

generator of $H^*(\mathbb{RP}^\infty, \mathbb{Z}/2) = \mathbb{Z}/2[t]$

$$\text{Hence } \bar{e}(\tilde{E}) \underset{\text{def}}{=:} w_0(E) t^n + w_1(E) t^{n-1} + \dots + w_n(E)$$

defines $w_i(E) \in H^i(B, \mathbb{Z}/2)$.

Prop.: These satisfy axioms of Stiefel-Whitney classes.

Proof:
- naturality: follows from naturality of the construction $E \mapsto \tilde{E}$ + naturality of Euler class
- product formula for $E \oplus F$: follows from $\widetilde{E \oplus F} \simeq \tilde{E} \oplus \tilde{F}$ and $\bar{e}(\tilde{E} \oplus \tilde{F}) = \bar{e}(\tilde{E}) \cup \bar{e}(\tilde{F})$
- normalization:

(1) $\to$ $w_0(E) = 1$: via naturality, enough to check for $B =$ point.

Then E is trivial, and $\tilde{E} \simeq \underbrace{\tau \oplus \dots \oplus \tau}_{n} \to \mathbb{RP}^\infty$

(since $\tau \to \mathbb{RP}^k$ obtained from trivial line bundle = normal bundle to $S^k \subset \mathbb{R}^{k+1}$ by quotienting by involution; take $\varinjlim$ as $k \to \infty$, and $\oplus$ n times).

Want to show $\bar{e}(\tau \oplus \dots \oplus \tau) = t^n$
By direct sum formula, enough to show $\bar{e}(\tau) = t \in H^1(\mathbb{RP}^\infty)$.
By naturality, this follows from nonvanishing for $\mathbb{RP}^1$.

(2) $w_1(\tau_{\mathbb{RP}^1})$ = generator: the t^0 component of $\bar{e}(\tilde{E}) \in H^n(\mathbb{RP}^1 \times \mathbb{RP}^\infty; \mathbb{Z}/2)$ is the restriction to $\mathbb{RP}^1 \times pt$, with $\bar{e}(\tilde{E})|_{\mathbb{RP}^1\times p} = \bar{e}(\tilde{E}|_{\mathbb{RP}^1\times p}) = \bar{e}(\tau)$ = generator. ∆

- By construction, we have $\bar{e}(E) = w_n(E) \in H^n(B, \mathbb{Z}/2)$.
- However, when E oriented, the $\mathbb{Z}/2$ Thom & Euler classes are images of oriented ones under change of coeff^t map $H^*(\cdot, \mathbb{Z}) \to H^*(\cdot, \mathbb{Z}/2)$.

<u>Corollary:</u> E oriented rank n $\Rightarrow$
The image of $e(E)$ under natural map $H^n(B,\mathbb{Z}) \to H^n(B, \mathbb{Z}/2)$ is $w_n(E)$.

<u>Euler & Stiefel-Whitney classes as obstructions:</u> (Milnor-Stasheff §12)

$E \to B$ rank n vect bundle / B CW-complex [optional, if E Euclidean; one retracts on other]

$\forall F$ fiber, $V_k(F)$ Stiefel manifold of (orthonormal) k-frames in F, $(n-k-1)$-connected (eg. $V_1(F) \underset{he}{\simeq} S^{n-1}$).

$\leadsto V_k(E) \to B$ fiber bundle of k-frames in fibers of E

<u>Q^n:</u> existence of a section of $V_k(E)$, ie. can we choose k pointwise linear indep^t sections of E?

Over the $(n-k)$-skeleton, no problem since $V_k(F)$ is $(n-k-1)$-connected: by induction, for each cell pullback of E is trivial, so problem of extending a given section over an r-cell is: given map $S^{r-1} \to V_k(\mathbb{R}^n)$, can it be extended over D^r; this is governed by $\pi_{r-1}(V_k(\mathbb{R}^n)) = 0$ if $r \le n-k$.

For each $(n-k+1)$-cell the obstruction lives in

$$\pi_{n-k}(V_k(\mathbb{R}^n)) \underset{\text{Steenrod}}{=} \begin{cases} \mathbb{Z}/2 & \text{for } n-k \text{ odd and } k>1 \\ \mathbb{Z} & \text{for } n-k \text{ even or } k=1 \end{cases}$$

(use fibrations $S^{n-k} \to V_k \to V_{k-1}$)

noncanonically, depends on orientation.

The obstruction is encoded by a cocycle $\in C^{n-k+1}(B, \{\pi_{n-k}(V_k(F))\})$ local coeff^t cohomology

(ie. by def., $(n-k+1)$-cell $\mapsto$ elt of $\pi_{n-k}(\dots)$) for a point $\in$ cell.

Changing choices over $(n-k)$-skeleton changes this cocycle by a coboundary.

$\to$ the actual invt is the <u>obstruction class</u> $\quad o_{n-k+1} \in H^{n-k+1}(B, \{\pi_{n-k}(V_k(F))\})$

Non manageable: $\forall$ parity, $\exists!$ nontrivial morphism $\pi_{n-k}(V_k(\mathbb{R}^n)) \to \mathbb{Z}/2$.

$\rightsquigarrow o_{n-k+1} \bmod 2 \in H^{n-k+1}(B, \mathbb{Z}/2)$.

Thm: $o_{n-k+1} \bmod 2 = w_{n-k+1}(E)$.

In particular: $w_1(E)$ = obstr. to finding section of $V_n(E)$ over 1-skeleton
= obstr. to orientability of E.

Indeed: ★ $\forall p \in$ 0-skeleton, choose elt $\varepsilon_p \in V_n(F_p) \leftrightarrow$ basis of F_p ($\Rightarrow$ orient. of F_p).
the obstr. 1-cocycle: given 1-cell $p \bullet\!\xrightarrow{\;e\;}\!\bullet\, q$ $\mapsto$ compare: do $\varepsilon_p, \varepsilon_q \in$ same conn. component of $V_n(\mathbb{R}^n)$?
ie. determine same orientation? can view this as $\in C^1(B, \mathbb{Z}/2)$
Changing choice at p affects cocycle for all 1-cells adjacent to p, ie. by coboundary of $(p \mapsto 1$, others $\mapsto 0)$.
so class $o_1 \in H^1(B, \mathbb{Z}/2)$ determines whether $\exists$ section of $V_n(E)$ over $B^{(1)}$ & also whether $E_{|B^{(1)}}$ orientable.

★ this suffices to determine orientability over all of B:
$\{$orientations of $F\} \simeq \{2$ pts$\}$ has no π_j for $j \geq 1$, so obstr. to extending over a $(j+1)$-cell given orientation on j-skeleton vanishes.

★ this agrees with $w_1(E)$: because obstruction class is natural wrt pullback, so only need to check for $\tau \to G_n(\mathbb{R}^\infty)$ where indeed obstr. to orientation (or basis) is determined by looking at unique 1-cell $= e_{\sigma=(1,\dots,n-1,n+1)}$
= loop of subspaces w/ basis $e_1, \dots, e_{n-1}$, (rotating unit vector in (e_n, e_{n+1}) plane)
orientation fails, obstr. cocycle = generator of $H^1(G_n(\mathbb{R}^\infty), \mathbb{Z}/2)$ ie. w_1.

- If E oriented rk n bundle, then $\pi_{n-1}(V_1(F)) \underset{\text{orient.}}{\simeq} \pi_{n-1}(S^{n-1}) \simeq \mathbb{Z}$ canonically,

so get $o_n \in H^n(B, \mathbb{Z})$

Thm: $o_n = e(E) \in H^n(B, \mathbb{Z})$ obstr. to a nowhere vanishing sec. of E over $B^{(n)}$.